Optimal Escape Paths

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A summary of Bellman's "Lost in a Forest" problem appears in [1]. Certain allied constants are described in [2, 3] and research is ongoing [4, 5, 6]. We will focus on just one facet of the problem for now, namely the following:

> *A hiker is lost in a forest whose shape is known to be a half-plane. What is the best path for him to follow to escape from the forest?*

This is equivalent to:

> *A swimmer is lost in a dense fog at sea, and she knows that the shore is a line. What is the best path for her to follow to search for the shore?*

Since no information is available concerning the initial distance or orientation of the boundary, a candidate path must be unbounded. Baeza-Yates, Culberson & Rawlins [7, 8, 9] claimed that the best path (which minimizes the maximum escape time) is a logarithmic spiral. Their argument was based on symmetry; a proof via the calculus of variations is still sought after [5, 6].

Speed is constant, thus escape time is proportional to arclength. If we assume that a logarithmic spiral $r = e^{\kappa\theta}$ is indeed optimal, then straightforward analysis leads to the best value of the parameter κ. Let the initial (unknown) distance from the boundary be R. Then the min-max logarithmic spiral can be shown to have parameter

$$\kappa = \tan\alpha = 0.2124695594... = \ln(1.2367284662...)$$

with arclength

$$R\csc\alpha\sec\beta = (13.8111351795...)R,$$

where α, β satisfy the simultaneous equations

$$\frac{1}{\tan\alpha} + \frac{1}{\tan\beta} = \frac{2\pi - \alpha - \beta}{\cos^2\alpha}, \qquad \frac{\cos\alpha}{\cos\beta} = e^{(2\pi-\alpha-\beta)\tan\alpha}.$$

It is surprising that such interesting constants emerge here, yet frustrating that a gap in the proof (for such a simple forest/sea) should persist.

0.1. Growth of Squares. While on the subject of logarithmic spirals, it seems natural to continue a discussion begun in [10]. Let $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $\ldots$ denote the Fibonacci sequence and $\varphi = (1 + \sqrt{5})/2$ denote the Golden mean. In the xy plane, draw the 1×1 square with center $(1/2, 1/2)$, then the adjacent 1×1 square with center $(-1/2, 1/2)$, then the adjacent 2×2 square with center $(0, -1)$, then the adjacent 3×3 square with center $(5/2, -1/2)$, then the adjacent 5×5 square with center $(3/2, 7/2)$, and so forth (in a counterclockwise manner). The n^{th} square is $f_n \times f_n$ and shares an edge between the two squares preceding it. Supposing we now translate the origin to the point $(2/5, 1/5)$, the logarithmic spiral $r = e^{\kappa\theta+\lambda}$ then asymptotically approaches the $f_n \times f_n$ square centers as $n \to \infty$, where [11]

$$\kappa = \frac{2}{\pi}\ln(\varphi) = 0.3063489625...,$$

$$\lambda_{\text{center}} = \frac{1}{2}\ln\left(\frac{\varphi+1}{10}\right) - \arctan(3)\kappa = -1.0527245979....$$

In the squares just constructed, consider instead the **leading vertices**

$$(0,1), \qquad (-1,0), \qquad (1,-2), \qquad (4,1), \qquad (-1,6), \qquad \ldots$$

and the **trailing vertices**

$$(1,1), \qquad (-1,1), \qquad (-1,-2), \qquad (4,-2), \qquad (4,6), \qquad \ldots$$

in the original coordinate system [11]. After translation (as before), the two associated asymptotic spirals possess the same κ but different λs:

$$\lambda_{\text{lead}} = \frac{1}{2}\ln\left(\frac{2(\varphi+2)}{25}\right) - \arctan(2\varphi-3)\kappa = -0.6909179135...,$$

$$\lambda_{\text{trail}} = \frac{1}{2}\ln\left(\frac{11\varphi+7}{25}\right) - \arctan(\varphi)\kappa = -0.3156737662....$$

There exists a nice duality between this material (starting with a square and concatenating) and earlier material (starting with a Golden rectangle and partitioning). In Figure 1.2 of [10], supposing we translate the origin to the point $((1+3\varphi)/5, (3-\varphi)/5)$, the spiral pictured there possesses the same κ but yet another λ:

$$\lambda'_{\text{lead}} = \frac{1}{2}\ln\left(\frac{2(\varphi+2)}{5}\right) - (\pi + \arctan(2\varphi-3))\,\kappa = -0.8486226074....$$

Other variations suggest themselves.

References

[1] S. R. Finch and J. E. Wetzel, Lost in a forest, *Amer. Math. Monthly* 11 (2004) 645-654.

[2] S. R. Finch, Moser's worm constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 491–497.

[3] S. R. Finch, Beam detection constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 515–519.

[4] S. R. Finch and J. A. Shonder, Lost at sea, math.OC/0411518.

[5] S. R. Finch and L.-Y. Zhu, Searching for a shoreline, math.OC/0501123.

[6] S. R. Finch, The logarithmic spiral conjecture, math.OC/0501133.

[7] R. A. Baeza-Yates, J. C. Culberson and G. J. E. Rawlins, Searching in the plane, *Inform. and Comput.* 106 (1993) 234–252; MR1241311 (94h:90019).

[8] R. A. Baeza-Yates, J. C. Culberson and G. J. E. Rawlins, Searching with uncertainty (extended abstract), *First Scandinavian Workshop on Algorithm Theory*, Proc. 1988 Halmstad conf., ed. R. Karlsson and A. Lingas, Lecture Notes in Comput. Sci. 318, Springer-Verlag, 1988, pp. 176–189.

[9] R. A. Baeza-Yates, J. C. Culberson and G. J. E. Rawlins, Searching with uncertainty (full report), Research Report CS–87–68, Dept. of Computer Sci., Univ. of Waterloo, 1987.

[10] S. R. Finch, The Golden mean, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 5–11.

[11] R. Fletcher, Investigation of the growth of squares, unpublished note (2004).