Variants of Brownian Motion

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We defined standard Brownian motion $\{W_t : t \geq 0\}$ in [1]. An alternative characterization of the Wiener process involves the limit of random walks. Let ε_1, ε_2, $\ldots$, ε_n be a sequence of independent identically distributed random variables, each possessing mean 0 and variance 1. Let

$$S_0 = 0, \qquad S_1 = \varepsilon_1, \qquad S_2 = \varepsilon_1 + \varepsilon_2, \qquad ..., \qquad S_n = \sum_{k=1}^{n} \varepsilon_k.$$

Then the random walk $\{S_k\}_{k=1}^n$ approaches Brownian motion on the unit interval in the sense that

$$\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \to W_t, \qquad 0 \leq t \leq 1$$

as $n \to \infty$, via the functional central limit theorem of Donsker [2, 3]. We are interested in the L_p-norm of Brownian motion

$$\|W\|_p = \begin{cases} \left(\int\limits_0^1 |W_t|^p \, dt \right)^{1/p} & \text{if } 0 < p < \infty, \\ \max\limits_{0 \leq t \leq 1} |W_t| & \text{if } p = \infty \end{cases}$$

for a number of reasons [4, 5]. Note that $\|W\|_p$ is itself a random variable. A distributional statement about $\|W\|_p$ hence translates into an asymptotic distributional statement about the l_p-norm of the random walk:

$$\mathrm{P}\left(\|W\|_p \leq x \right) = \begin{cases} \lim\limits_{n \to \infty} \mathrm{P}\left(\left(\sum\limits_{k=1}^{n} |S_k|^p \right)^{1/p} \leq n^{\frac{1}{2}+\frac{1}{p}} x \right) & \text{if } 0 < p < \infty, \\ \lim\limits_{n \to \infty} \mathrm{P}\left(\max\left\{ |S_1|, |S_2|, \ldots, |S_n| \right\} \leq n^{1/2} x \right) & \text{if } p = \infty. \end{cases}$$

In the following sections, we will discuss the cases $p = \infty$, 1 and 2 for several variants of Brownian motion. Corresponding problems for all other values of $p > 0$ remain unsolved.

Some preliminary definitions include

$$\delta_m = \frac{\Gamma(m+\frac{1}{2})}{\sqrt{\pi}m!} = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots(2m)} & \text{if } m \geq 1, \\ 1 & \text{if } m = 0, \end{cases}$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt = 1 - \operatorname{erfc}(x),$$

$$\operatorname{Ai}(x) = \begin{cases} \frac{1}{3}(-x)^{1/2}\left[J_{-\frac{1}{3}}\left(\frac{2}{3}(-x)^{3/2}\right) + J_{\frac{1}{3}}\left(\frac{2}{3}(-x)^{3/2}\right)\right] & \text{if } x < 0, \\ \frac{1}{3}x^{1/2}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right)\right] & \text{if } x \geq 0, \end{cases}$$

$$K_{\frac{1}{4}}(x) = \frac{\pi}{\sqrt{2}}\left[I_{-\frac{1}{4}}(x) - I_{\frac{1}{4}}(x)\right]$$

where $J_\nu(x)$ and $I_\nu(x)$ are the well-known Bessel functions. Also, for $x > 0$ and $0 < a < b$, let

$$U(a,b,x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-tx}t^{a-1}(1+t)^{b-a-1}dt.$$

This is called the confluent hypergeometric function of the second kind (in contrast to [1]). Finally, define the Riemann xi function

$$\xi(z) = \tfrac{1}{2}z(z-1)\pi^{-z/2}\Gamma(\tfrac{1}{2}z)\zeta(z), \qquad \operatorname{Re}(z) > 1,$$

which serves as a tantalizing link between Brownian motion and number theory [6]. This can be analytically continued to an entire function via functional equation $\xi(z) = \xi(1-z)$.

0.1. Bridge. A Brownian bridge $\{X_t : 0 \leq t \leq 1\}$ has the same distribution as $\{W_t : 0 \leq t \leq 1\}$, conditioned on $W_1 = 0$. The maximum of $|X_t|$ turns out to be closely allied with the Kolmogorov-Smirnov goodness-of-fit test [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]:

$$\mathrm{P}\left(\|X\|_\infty \leq x\right) = \sum_{k=-\infty}^{\infty}(-1)^k e^{-2k^2x^2} = \frac{\sqrt{2\pi}}{x}\sum_{k=0}^{\infty} e^{-\pi^2(2k+1)^2/(8x^2)}$$

(and the right-hand equality follows via Poisson summation). This distribution has moments

$$\mathrm{E}\left(\|X\|_\infty\right) = \sqrt{\frac{\pi}{2}}\ln(2), \qquad \mathrm{E}\left(\|X\|_\infty^2\right) = \frac{\pi^2}{12}$$

and median 0.8275735551.... It also satisfies [17, 18]

$$\mathrm{E}\left(\|X\|_\infty^z\right) = 2\frac{1-2^{1-z}}{z-1}\left(\frac{\pi}{2}\right)^{z/2}\xi(z)$$

for all complex z.

Takács [19, 20], building on Cifarelli [21], Shepp [22], Rice [23] and Johnson & Killeen [24], computed that

$$\mathrm{P}\left(\|X\|_1 \leq x\right) = \frac{\sqrt{\pi}}{18^{1/6}x}\sum_{j=1}^{\infty} e^{-u_j}u_j^{-1/3}\,\mathrm{Ai}\left((3u_j/2)^{2/3}\right)$$

for $x > 0$, where $u_j = (a_j')^3/(27x^2)$ and $0 < a_1' < a_2' < \ldots$ are the zeroes [25] of $\mathrm{Ai}'(-x)$. This distribution has moments

$$\mathrm{E}\left(\|X\|_1\right) = \frac{1}{4}\sqrt{\frac{\pi}{2}}, \qquad \mathrm{E}\left(\|X\|_1^2\right) = \frac{7}{60}$$

and median 0.2817802658....

Anderson & Darling [26, 27, 28, 29], building on Smirnov [30], obtained that

$$\mathrm{P}\left(\|X\|_2^2 \leq x\right) = \frac{1}{\pi\sqrt{x}}\sum_{j=0}^{\infty}\sqrt{4j+1}e^{-(4j+1)^2/(16x)}\delta_j K_{1/4}\left((4j+1)^2/(16x)\right),$$

which has moments

$$\mathrm{E}\left(\|X\|_2^2\right) = \frac{1}{6}, \qquad \mathrm{E}\left(\|X\|_2^4\right) = \frac{1}{20}$$

and median 0.1188795509.... The L_2-norm, squared, of X_t turns out to be closely allied with the Cramér-von Mises goodness-of-fit test [31, 32, 33].

0.2. Excursion. A Brownian excursion $\{Y_t : 0 \leq t \leq 1\}$ has the same distribution as $\{W_t : 0 \leq t \leq 1\}$, conditioned on $W_t > 0$ for all $0 < t < 1$ and $W_1 = 0$.

Chung [34, 35], Kennedy [36] and Durrett & Iglehart [37, 38] showed that

$$\mathrm{P}\left(\|Y\|_\infty \leq x\right) = \sum_{k=-\infty}^{\infty}(1-4k^2x^2)e^{-2k^2x^2} = \frac{\sqrt{2}\pi^{5/2}}{x^3}\sum_{k=1}^{\infty}k^2e^{-\pi^2k^2/(2x^2)},$$

which has moments

$$\mathrm{E}\left(\|Y\|_\infty\right) = \sqrt{\frac{\pi}{2}}, \qquad \mathrm{E}\left(\|Y\|_\infty^2\right) = \frac{\pi^2}{6},$$

median 1.2234880197..., and also satisfies [17, 18]

$$\mathrm{E}\left(\|Y\|_\infty^z\right) = 2\left(\frac{\pi}{2}\right)^{z/2}\xi(z)$$

for all complex z.

Takács [19, 39], building on Getoor & Sharpe [40], Darling [41], Louchard [42, 43] and Groenboom [44], obtained that

$$\mathrm{P}\left(\|Y\|_1 \le x\right) = \frac{\sqrt{6}}{x}\sum_{j=1}^{\infty} e^{-v_j} v_j^{2/3} U\left(\tfrac{1}{6}, \tfrac{4}{3}, v_j\right)$$

for $x > 0$, where $v_j = 2a_j^3/(27x^2)$ and $0 < a_1 < a_2 < \ldots$ are the zeroes [25] of $\mathrm{Ai}(-x)$. This distribution has moments

$$\mathrm{E}\left(\|Y\|_1\right) = \sqrt{\frac{\pi}{8}}, \qquad \mathrm{E}\left(\|Y\|_1^2\right) = \frac{5}{12}$$

and median 0.6070363869....

The L_2 case seems to be open for Brownian excursion.

0.3. Meander. A Brownian meander $\{Z_t : 0 \le t \le 1\}$ has the same distribution as $\{W_t : 0 \le t \le 1\}$, conditioned on $W_t > 0$ for all $0 < t < 1$. Note that Z_1 need not be zero.

Durrett & Iglehart [37, 38] computed that

$$\mathrm{P}\left(\|Z\|_\infty \le x\right) = \sum_{k=-\infty}^{\infty}(-1)^k e^{-k^2x^2/2} = \frac{2^{3/2}\sqrt{\pi}}{x}\sum_{k=0}^{\infty} e^{-\pi^2(2k+1)^2/(2x^2)}.$$

Observe that the distribution of $\|Z\|_\infty$ is the same as the distribution of $2\|X\|_\infty$. Hence it has moments

$$\mathrm{E}\left(\|Z\|_\infty\right) = \sqrt{2\pi}\ln(2), \qquad \mathrm{E}\left(\|Z\|_\infty^2\right) = \frac{\pi^2}{3}$$

and median 1.6551471103...

Takács [45] proved that

$$\mathrm{P}\left(\|Z\|_1 \le x\right) = \frac{\sqrt{\pi}}{18^{1/6}x}\sum_{j=1}^{\infty} b_j e^{-\tilde{v}_j}\tilde{v}_j^{-1/3}\,\mathrm{Ai}\left((3\tilde{v}_j/2)^{2/3}\right)$$

for $x > 0$, where $\tilde{v}_j = v_j/2$ and v_j, a_j are as before, and where

$$b_j = \frac{a_j}{3\,\mathrm{Ai}'(-a_j)}\left(1 + 3\int_0^{a_j}\mathrm{Ai}(-s)\,ds\right).$$

This distribution has moments

$$\mathrm{E}\left(\|Z\|_1\right) = \frac{3}{4}\sqrt{\frac{\pi}{2}}, \qquad \mathrm{E}\left(\|Z\|_1^2\right) = \frac{59}{60}$$

and median 0.8900420723....

The L_2 case seems to be open for Brownian meander.

0.4. Motion. We return to Brownian motion. Erdös & Kac [46, 47, 48, 49] computed that

$$\begin{aligned}
\mathrm{P}\left(\|W\|_\infty \le x\right) &= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}e^{-\pi^2(2k+1)^2/(8x^2)} \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty}(-1)^k\left[\operatorname{erf}\left(\tfrac{(2k+1)x}{\sqrt{2}}\right) - \operatorname{erf}\left(\tfrac{(2k-1)x}{\sqrt{2}}\right)\right] \\
&= -1 + \sum_{k=-\infty}^{\infty}\left[\operatorname{erf}\left(\tfrac{(4k+1)x}{\sqrt{2}}\right) - \operatorname{erf}\left(\tfrac{(4k-1)x}{\sqrt{2}}\right)\right],
\end{aligned}$$

which has moments

$$\mathrm{E}\left(\|W\|_\infty\right) = \sqrt{\frac{\pi}{2}}, \qquad \mathrm{E}\left(\|W\|_\infty^2\right) = 2G$$

and median 1.1489732581.... This is a remarkable appearance of Catalan's constant G!

Takács [50, 51], building on Kac [52] and Schwinger [53], found that

$$\mathrm{P}\left(\|W\|_1 \le y\right) = \sqrt{\frac{3}{\pi}}\int_0^y \frac{1}{x}\sum_{j=1}^{\infty} c_j e^{-\tilde{u}_j}\tilde{u}_j^{2/3}U\left(\tfrac{1}{6},\tfrac{4}{3},\tilde{u}_j\right)dx$$

for $y > 0$, where $\tilde{u}_j = 2u_j$ and u_j, a_j' are as before, and where

$$c_j = \frac{1}{3a_j'\,\mathrm{Ai}(-a_j')}\left(1 + 3\int_0^{a_j'}\mathrm{Ai}(-s)\,ds\right).$$

This distribution has moments

$$\mathrm{E}\left(\|W\|_1\right) = \frac{4}{3}\frac{1}{\sqrt{2\pi}}, \qquad \mathrm{E}\left(\|W\|_1^2\right) = \frac{3}{8}$$

and median 0.4510953819.... We wonder whether the integral for $\mathrm{P}(\|W\|_1 \leq y)$ can be termwise integrated.

Cameron & Martin [46, 54, 55, 56, 57] proved that

$$\mathrm{P}\left(\|W\|_2^2 \leq x\right) = \sqrt{2}\sum_{j=0}^{\infty}(-1)^j \delta_j \operatorname{erfc}\left(\tfrac{4j+1}{2\sqrt{2x}}\right)$$

which has moments

$$\mathrm{E}\left(\|W\|_2^2\right) = \frac{1}{2}, \qquad \mathrm{E}\left(\|W\|_2^4\right) = \frac{7}{12},$$

median 0.2904760595... and Laplace transform

$$\mathrm{E}\left(\exp(-\lambda\,\|W\|_2^2)\right) = \sqrt{\sec\left(\sqrt{-2\lambda}\right)}.$$

We close with several unanswered questions. Define the positive part of W_t to be $W_t^+ = \max\{W_t, 0\}$. Perman & Wellner [58, 59] studied the 1-norm of W_t^+ and found the following double Laplace transform:

$$\int_0^{\infty} e^{-\mu\lambda}\,\mathrm{E}\left\{\exp\left(-\sqrt{2}\lambda^{3/2}\left\|W^+\right\|_1\right)\right\} d\lambda = \frac{\mu^{-1/2}\operatorname{Ai}(\mu) + \frac{1}{3} - \int_0^{\mu}\operatorname{Ai}(s)\,ds}{\sqrt{\mu}\operatorname{Ai}(\mu) - \operatorname{Ai}'(\mu)}$$

as well as moments:

$$\mathrm{E}\left(\|W^+\|_1\right) = \frac{2}{3}\frac{1}{\sqrt{2\pi}}, \qquad \mathrm{E}\left(\|W^+\|_1^2\right) = \frac{17}{96}.$$

Does an explicit formula for $\mathrm{P}\left(\|W^+\|_1 \leq x\right)$ exist? What can be said for other values of $p > 0$?

Brownian motion with drift (of linear type $W_t + \alpha t$ or parabolic type $W_t - \beta t^2$) would be interesting to report on later [60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73]. Of all possible issues, we examine just two. When analyzing $W_t + \alpha t$ for $\alpha > 0$, is the formula [63]

$$\int_0^{\pi/2} \frac{\exp(-x\cot(x))\sin(x)}{1+\exp(-\pi\cot(x))}\,dx = \int_0^{\infty}\left[\frac{1}{2} - \exp(-y\coth(y))\sinh(y)\right] dy$$

valid? Numerics suggest yes; a rigorous proof would be good to see someday. The expected maximum value of $W_t - (1/2)t^2$ is [72]

$$\frac{2^{-1/3}}{2\pi i}\int_{-\infty}^{\infty} \frac{z}{\operatorname{Ai}(iz)^2}\,dz = 0.9961930199...$$

(among several integral expressions) and we wonder if similar formulas exist for higher moments.

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