Cubic and Quartic Characters

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June 5, 2009

In this essay, we revisit Dirichlet characters [1], but focusing here on non-real cases (that is, of order exceeding 2).

Let $\mathbb{Z}_n^*$ denote the group (under multiplication modulo n) of integers relatively prime to n, and let $\mathbb{C}^*$ denote the group (under ordinary multiplication) of nonzero complex numbers. We wish to examine homomorphisms $\chi : \mathbb{Z}_n^* \to \mathbb{C}^*$ satisfying certain requirements. A Dirichlet character χ is **quadratic** if $\chi(k)^2 = 1$ for every k in $\mathbb{Z}_n^*$. It is well-known that, if $\chi \neq 1$ is a primitive quadratic character modulo n, then $D = \chi(-1)n$ is a fundamental discriminant and

$$\chi(k) = \left(\frac{D}{k}\right) \qquad \text{for all } k \in \mathbb{Z}_n^*$$

where (D/k) is the Kronecker-Jacobi-Legendre symbol. A character χ is real if and only if it is quadratic. By the correspondence with $(D/.)$, quadratic characters can be said to be completely understood.

A Dirichlet character χ is **cubic** if $\chi(k)^3 = 1$ for every k in $\mathbb{Z}_n^*$. Let $\omega = (-1 + i\sqrt{3})/2$ where i is the imaginary unit. Let $a + b\omega$ be a prime in the ring $\mathbb{Z}[\omega]$ of Eisenstein-Jacobi integers with norm $a^2 - ab + b^2 \neq 3$. For any positive integer n in $\mathbb{Z}$, define the cubic residue symbol [2, 3]

$$\left(\frac{n}{a + b\omega}\right)_3$$

to be 0 if n is divisible by $a + b\omega$; otherwise it is the unique power ω^j for $0 \leq j \leq 2$ such that

$$n^{(a^2-ab+b^2-1)/3} \equiv \omega^j \operatorname{mod}(a + b\omega).$$

The only prime divisor of 9 is $1 - \omega$, which has norm 3. Hence we will need an alternative way of representing characters:

$$f_q(n,k) = \begin{cases} \omega^e & \text{if } n \equiv k^e \operatorname{mod} q \\ 0 & \text{otherwise} \end{cases}$$

especially in the case $q = 9$. The first several cubic characters are

$$f_7(n,5) = \left(\tfrac{n}{2+3\omega}\right)_3\Big|_{n=1,\ldots,7} = \{1,\omega,\omega^2,\omega^2,\omega,1,0\},$$

$$f_7(n,3) = \left(\tfrac{n}{-1-3\omega}\right)_3\Big|_{n=1,\ldots,7} = \{1,\omega^2,\omega,\omega,\omega^2,1,0\},$$

$$f_9(n,2)|_{n=1,\ldots,9} = \{1,\omega,0,\omega^2,\omega^2,0,\omega,1,0\},$$

$$f_9(n,5)|_{n=1,\ldots,9} = \{1,\omega^2,0,\omega,\omega,0,\omega^2,1,0\},$$

$$f_{13}(n,2) = \left(\tfrac{n}{-4-3\omega}\right)_3\Big|_{n=1,\ldots,13} = \{1,\omega,\omega,\omega^2,1,\omega^2,\omega^2,1,\omega^2,\omega,\omega,1,0\},$$

$$f_{13}(n,6) = \left(\tfrac{n}{-1+3\omega}\right)_3\Big|_{n=1,\ldots,13} = \{1,\omega^2,\omega^2,\omega,1,\omega,\omega,1,\omega,\omega^2,\omega^2,1,0\},$$

$$f_{19}(n,2) = \left(\tfrac{n}{2-3\omega}\right)_3\Big|_{n=1,\ldots,19} = \{1,\omega,\omega,\omega^2,\omega,\omega^2,1,1,\omega^2,\omega^2,1,1,\omega^2,\omega,\omega^2,\omega,\omega,1,0\},$$

$$f_{19}(n,10) = \left(\tfrac{n}{5+3\omega}\right)_3\Big|_{n=1,\ldots,19} = \{1,\omega^2,\omega^2,\omega,\omega^2,\omega,1,1,\omega,\omega,1,1,\omega,\omega^2,\omega,\omega^2,\omega^2,1,0\},$$

$$\begin{aligned} f_{31}(n,3) &= \left(\tfrac{n}{5+6\omega}\right)_3\Big|_{n=1,\ldots,31} \\ &= \{1,1,\omega,1,\omega^2,\omega,\omega,1,\omega^2,\omega^2,\omega^2,\omega,\omega^2,\omega,1,1,\omega,\omega^2,\omega,\omega^2, \\ &\quad \omega^2,\omega^2,1,\omega,\omega,\omega^2,1,\omega,1,1,0\}, \end{aligned}$$

$$\begin{aligned} f_{31}(n,11) &= \left(\tfrac{n}{-1-6\omega}\right)_3\Big|_{n=1,\ldots,31} \\ &= \{1,1,\omega^2,1,\omega,\omega^2,\omega^2,1,\omega,\omega,\omega,\omega^2,\omega,\omega^2,1,1,\omega^2,\omega,\omega^2,\omega, \\ &\quad \omega,\omega,1,\omega^2,\omega^2,\omega,1,\omega^2,1,1,0\}, \end{aligned}$$

$$\begin{aligned} f_{37}(n,2) &= \left(\tfrac{n}{-4+3\omega}\right)_3\Big|_{n=1,\ldots,37} \\ &= \{1,\omega,\omega^2,\omega^2,\omega^2,1,\omega^2,1,\omega,1,1,\omega,\omega^2,1,\omega,\omega,\omega,\omega^2,\omega^2,\omega, \\ &\quad \omega,\omega,1,\omega^2,\omega,1,1,\omega,1,\omega^2,1,\omega^2,\omega^2,\omega^2,\omega,1,0\}, \end{aligned}$$

$$\begin{aligned} f_{37}(n,5) &= \left(\tfrac{n}{-7-3\omega}\right)_3\Big|_{n=1,\ldots,37} \\ &= \{1,\omega^2,\omega,\omega,\omega,1,\omega,1,\omega^2,1,1,\omega^2,\omega,1,\omega^2,\omega^2,\omega^2,\omega,\omega,\omega^2, \\ &\quad \omega^2,\omega^2,1,\omega,\omega^2,1,1,\omega^2,1,\omega,1,\omega,\omega,\omega,\omega^2,1,0\}. \end{aligned}$$

A Dirichlet character χ is **quartic** (**biquadratic**) if $\chi(k)^4 = 1$ for every k in $\mathbb{Z}_n^*$. Let $a + bi$ be a prime in the ring $\mathbb{Z}[i]$ of Gaussian integers with norm $a^2 + b^2 \neq 2$. For any positive integer n in $\mathbb{Z}$, define the quartic (biquadratic) residue symbol [2, 3]

$$\left(\frac{n}{a+bi}\right)_4$$

to be 0 if n is divisible by $a + bi$; otherwise it is the unique power i^j for $0 \leq j \leq 3$ such that

$$n^{(a^2+b^2-1)/4} \equiv i^j \bmod (a + bi).$$

The only prime divisor of 16 is $1+i$, which has norm 2. We will again need alternative ways of representing characters:

$$f_q(n,k) = \begin{cases} i^e & \text{if } n \equiv k^e \bmod q, \\ 0 & \text{otherwise,} \end{cases}$$

$$g_q(n,k) = \begin{cases} i^e & \text{if } n \equiv k^e \bmod q \text{ or } q - n \equiv k^e \bmod q, \\ 0 & \text{otherwise,} \end{cases}$$

$$h_q(n,k,\ell,m) = \begin{cases} i^e & \text{if } n \equiv k^e \bmod q \text{ or } n \equiv \ell^e \bmod q, \\ (-1)^{e+1} & \text{if } q - n \equiv m^e \bmod q, \\ 0 & \text{otherwise} \end{cases}$$

especially in the cases $q = 15, 16, 20$ and 35. The first several non-real quartic characters are

$$f_5(n,2) = \left(\tfrac{n}{-1-2i}\right)_4\Big|_{n=1,\ldots,5} = \{1, i, -i, -1, 0\},$$

$$f_5(n,3) = \left(\tfrac{n}{-1+2i}\right)_4\Big|_{n=1,\ldots,5} = \{1, -i, i, -1, 0\},$$

$$f_{13}(n,2) = \left(\tfrac{n}{3-2i}\right)_4\Big|_{n=1,\ldots,13} = \{1, i, 1, -1, i, i, -i, -i, 1, -1, -i, -1, 0\},$$

$$f_{13}(n,7) = \left(\tfrac{n}{3+2i}\right)_4\Big|_{n=1,\ldots,13} = \{1, -i, 1, -1, -i, -i, i, i, 1, -1, i, -1, 0\},$$

$$g_{15}(n,2)|_{n=1,\ldots,15} = \{1, i, 0, -1, 0, 0, -i, -i, 0, 0, -1, 0, i, 1, 0\},$$

$$g_{15}(n,8)|_{n=1,\ldots,15} = \{1, -i, 0, -1, 0, 0, i, i, 0, 0, -1, 0, -i, 1, 0\},$$

$$g_{16}(n,3)|_{n=1,\ldots,16} = \{1, 0, i, 0, -i, 0, -1, 0, -1, 0, -i, 0, i, 0, 1, 0\},$$

$$g_{16}(n,5)|_{n=1,\ldots,16} = \{1, 0, -i, 0, i, 0, -1, 0, -1, 0, i, 0, -i, 0, 1, 0\},$$

$$h_{16}(n,3,5,9)|_{n=1,\ldots,16} = \{1, 0, i, 0, i, 0, 1, 0, -1, 0, -i, 0, -i, 0, -1, 0\},$$

$$h_{16}(n,11,13,9)|_{n=1,\ldots,16} = \{1,0,-i,0,-i,0,1,0,-1,0,i,0,i,0,-1,0\},$$
$$f_{17}(n,3) = \left(\tfrac{n}{1-4i}\right)_4\Big|_{n=1,\ldots,17} = \{1,-1,i,1,i,-i,-i,-1,-1,-i,-i,i,1,i,-1,1,0\},$$
$$f_{17}(n,6) = \left(\tfrac{n}{1+4i}\right)_4\Big|_{n=1,\ldots,17} = \{1,-1,-i,1,-i,i,i,-1,-1,i,i,-i,1,-i,-1,1,0\},$$
$$g_{20}(n,3)|_{n=1,\ldots,20} = \{1,0,i,0,0,0,-i,0,-1,0,-1,0,-i,0,0,0,i,0,1,0\},$$
$$g_{20}(n,7)|_{n=1,\ldots,20} = \{1,0,-i,0,0,0,i,0,-1,0,-1,0,i,0,0,0,-i,0,1,0\},$$

$$\begin{aligned} f_{29}(n,2) &= \left(\tfrac{n}{-5-2i}\right)_4\Big|_{n=1,\ldots,29} \\ &= \{1,i,i,-1,-1,-1,1,-i,-1,-i,i,-i,-1,i,-i,1,i,-i,i,1, \\ &\quad i,-1,1,1,1,-i,-i,-1,0\}, \end{aligned}$$

$$\begin{aligned} f_{29}(n,8) &= \left(\tfrac{n}{-5+2i}\right)_4\Big|_{n=1,\ldots,29} \\ &= \{1,-i,-i,-1,-1,-1,1,i,-1,i,-i,i,-1,-i,i,1,-i,i,-i,1, \\ &\quad -i,-1,1,1,1,i,i,-1,0\}, \end{aligned}$$

$$\begin{aligned} g_{35}(n,2)|_{n=1,\ldots,35} &= \{1,i,i,-1,0,-1,0,-i,-1,0,1,-i,i,0,0,1,-i,-i,1,0, \\ &\quad 0,i,-i,1,0,-1,-i,0,-1,0,-1,i,i,1,0\}, \end{aligned}$$

$$\begin{aligned} g_{35}(n,18)|_{n=1,\ldots,35} &= \{1,-i,-i,-1,0,-1,0,i,-1,0,1,i,-i,0,0,1,i,i,1,0, \\ &\quad 0,-i,i,1,0,-1,i,0,-1,0,-1,-i,-i,1,0\}, \end{aligned}$$

$$\begin{aligned} f_{37}(n,2) &= \left(\tfrac{n}{-1+6i}\right)_4\Big|_{n=1,\ldots,37} \\ &= \{1,i,-1,-1,-i,-i,1,-i,1,1,-1,1,-i,i,i,1,-i,i,-i,i, \\ &\quad -1,-i,-i,i,-1,1,-1,-1,i,-1,i,i,1,1,-i,-1,0\}, \end{aligned}$$

$$\begin{aligned} f_{37}(n,5) &= \left(\tfrac{n}{-1-6i}\right)_4\Big|_{n=1,\ldots,37} \\ &= \{1,-i,-1,-1,i,i,1,i,1,1,-1,1,i,-i,-i,1,i,-i,i,-i, \\ &\quad -1,i,i,-i,-1,1,-1,-1,-i,-1,-i,-i,1,1,i,-1,0\}. \end{aligned}$$

We mention that [4]

$$\begin{array}{l}\text{\# Dirichlet characters of} \\ \text{order } \ell \text{ and modulus } n\end{array} = \begin{array}{l}\text{\# solutions } x \text{ in } \mathbb{Z}_n^* \text{ of} \\ \text{the equation } x^\ell = 1\end{array}$$

and thus, by Möbius inversion,

$$\begin{array}{l}\#\text{ primitive quadratic Dirichlet}\\ \text{characters of modulus } \leq N\end{array} \sim \frac{6}{\pi^2}N,$$

$$\begin{array}{l}\#\text{ primitive cubic Dirichlet}\\ \text{characters of modulus } \leq N\end{array} \sim A\,N,$$

$$\begin{array}{l}\#\text{ primitive quartic Dirichlet}\\ \text{characters of modulus } \leq N\end{array} \sim B\,N\ln(N),$$

as $N \to \infty$, where [5, 6, 7]

$$A = \frac{11\sqrt{3}}{18\pi}\prod_{p\equiv 1 \bmod 3}\left(1-\frac{2}{p(p+1)}\right) = 0.3170565167...,$$

$$B = \frac{7}{\pi}\frac{1}{16K^2}\prod_{p\equiv 1 \bmod 4}\left(1-\frac{5p-3}{p^2(p+1)}\right) = 0.1908767211...$$

and K is the Landau-Ramanujan constant [8]. No one appears to have examined B before.

Now define the **Dirichlet L-series associated to** $\chi \neq 1$:

$$L_\chi(z) = \sum_{n=1}^{\infty}\chi(n)n^{-z} = \prod_p\left(1-\chi(p)p^{-z}\right)^{-1}, \qquad \mathrm{Re}(z) > 1$$

which can be made into an entire function. Special values are more complicated for cubic/quartic characters than for quadratic characters [1]. For example, if $\chi = (\cdot/(2+3\omega))_3$, then

$$L_\chi(1) = 7^{-2/3}(-2-3\omega)^{1/3}\left(\omega^2\ln(y_1)+\omega\ln(y_2)+\ln(y_3)\right)$$

where $y_1 < y_2 < y_3$ are the (real) zeroes of $y^3-7y^2+14y-7$; if $\chi = f_9(\cdot,2)$, then

$$L_\chi(1) = -\tfrac{2}{3}\omega^{1/3}\left(\omega^2\ln\left(\sin\left(\tfrac{2\pi}{9}\right)\right)+\omega\ln\left(\cos\left(\tfrac{\pi}{18}\right)\right)+\ln\left(\sin\left(\tfrac{\pi}{9}\right)\right)\right).$$

As more examples, if $\chi = (\cdot/(-1-2i))_4$, then

$$L_\chi(1) = 2^{1/2}5^{-5/4}(3+4i)^{1/4}\pi;$$

if $\chi = g_{16}(\cdot,3)$, then

$$L_\chi(1) = -\tfrac{1}{2}i^{1/4}\left(i\ln\left(\cot\left(\tfrac{3\pi}{16}\right)\right)+\ln\left(\tan\left(\tfrac{\pi}{16}\right)\right)\right);$$

if $\chi = h_{16}(\cdot, 3, 5, 9)$, then

$$L_\chi(1) = 8^{-1/2} i^{1/4} \pi.$$

See a general treatment of quartic cases in [9].

The elaborate formulas for moments of $L_\chi(1/2)$ over primitive quadratic characters χ do not yet appear to have precise analogs for primitive cubic characters. Baier & Young [10] proved that

$$\sum_{q \le Q} \sum_{\chi} |L_\chi(1/2)|^2 = O\left(Q^{6/5+\varepsilon}\right)$$

as $Q \to \infty$, for any $\varepsilon > 0$, where the big-O constant depends on ε. The inner summation is over all primitive cubic characters modulo q. As a consequence, $L_\chi(1/2) \neq 0$ for infinitely many such χ.

0.1. Cubic Twists. Given an elliptic curve E over $\mathbb{Q}$ with L-series

$$L_E(z) = \sum_{n=1}^{\infty} c_n n^{-z},$$

the L-series obtained via twisting $L_E(z)$ by a cubic character χ is

$$L_{E,\chi}(z) = \sum_{n=1}^{\infty} \chi(n) c_n n^{-z}.$$

Of course, while each $c_n \in \mathbb{Z}$, the coefficients $\chi(n)c_n \in \mathbb{Z}[\omega]$ need not be real. This generalizes the sense of quadratic twists discussed in [11]; we refer to a paper of David, Fearnley & Kisilevsky [6] for more information on such L-series.

There is a different sense of cubic twists that interests us – it is important for the study of the family of elliptic curves F_d given by $x^3 + y^3 = d$ – and features the cubic residue symbol $(d/\cdot)_3$ in an intriguing way. We mentioned the problem of evaluating $L_{F_d}(1)$ for cube-free $d > 2$ in [11] but did not give details. By definition [12],

$$\begin{aligned} L_{F_d}(z) &= \sum_{\substack{a,b \in \mathbb{Z} \\ a \equiv 1 \bmod 3 \\ b \equiv 0 \bmod 3}} (a + b\omega^2) \left(\frac{d}{a + b\omega}\right)_3 (a^2 - ab + b^2)^{-z} \\ &= \sum_{\substack{a,b \in \mathbb{Z} \\ a \equiv 1 \bmod 3 \\ b \equiv 0 \bmod 3}} (a + b\omega) \left(\frac{d}{a + b\omega^2}\right)_3 (a^2 - ab + b^2)^{-z} \\ &= \prod_{p \equiv 2 \bmod 3} (1 + p^{1-2z})^{-1} \cdot \prod_{p \equiv 1 \bmod 3} (1 - c_p p^{-z} + p^{1-2z})^{-1} \end{aligned}$$

where

$$c_p = (h + k\omega^2)\left(\frac{d}{h + k\omega}\right)_3 + (h + k\omega)\left(\frac{d}{h + k\omega^2}\right)_3$$

and $p = (h+k\omega)(h+k\omega^2)$, $h \equiv 1 \bmod 3$, $k \equiv 0 \bmod 3$. To extend to composite indices, use the usual recurrence $c_{p^j} = c_{p^{j-1}} c_p - p\, c_{p^{j-2}}$ for $j \geq 2$, $c_1 = 1$ and $c_{m\,n} = c_m c_n$ for coprime integers m, n.

For $d = 1$ and $p \equiv 1 \bmod 3$, it is known that $c_p = \gamma_p$, where γ_p is the unique integer $\alpha \equiv 2 \bmod 3$ such that $\alpha^2 + 3\beta^2 = 4p$ for some integer $\beta \equiv 0 \bmod 3$. Now, for $d > 1$ and $p \equiv 1 \bmod 3$, $p \nmid d$, it can be shown that c_p is the unique integer $\alpha \equiv 2 \bmod 3$ such that three conditions:

- $\alpha^2 + 3\beta^2 = 4p$ for some integer β
- $\alpha \equiv d^{(p-1)/3}\gamma_p \bmod p$
- $|\alpha| < 2\sqrt{p}$

are simultaneously satisfied [13].

Sextic twists are required to study Bachet's equation $y^2 = x^3 + n$ for arbitrary n (the Fermat cubic problem is a special case with $n = -432d^2$ and d cube-free). Such residue symbols are beyond us. Here is a formula for L-series coefficients c_p in this more general setting: when $p = 3$, $p|n$ or $p \equiv 2 \bmod 3$, we have $c_p = 0$; otherwise [14]

$$c_p = \left(\frac{n}{p}\right) \cdot \begin{cases} 2a - b & \text{if } (4n)^{(p-1)/3} \equiv 1 \bmod p, \\ -a - b & \text{if } (4n)^{(p-1)/3} b \equiv -a \bmod p, \\ 2b - a & \text{if } (4n)^{(p-1)/3} a \equiv -b \bmod p \end{cases}$$

where $p = a^2 - ab + b^2$ with $a \equiv 1 \bmod 3$, $b \equiv 0 \bmod 3$ and $(\cdot/\cdot)$ is the Kronecker-Jacobi-Legendre symbol. The sequence of integers for which $y^2 = x^3 + n$ has zero rank [15]:

$$..., -12, -10, -9, -8, -6, -5, -3, -1, 1, 4, 6, 7, 13, 14, 16, 20, ...$$

deserves close attention!

0.2. Quartic Twists. Quartic twists are required to study $y^2 = x^3 - n\,x$ for arbitrary n (the congruent number problem is a special case with $n = d^2$ and d

square-free [11]). Analogous to the expression for $L_{F_d}(z)$,

$$\begin{aligned} L_{E_n}(z) &= \sum_{\substack{a,b\in\mathbb{Z} \\ a\equiv 1 \bmod 4 \\ b\equiv 0 \bmod 2}} (a-bi)\left(\frac{-n}{a+bi}\right)_4 (a^2+b^2)^{-z} \\ &= \sum_{\substack{a,b\in\mathbb{Z} \\ a\equiv 1 \bmod 4 \\ b\equiv 0 \bmod 2}} (a+bi)\left(\frac{-n}{a-bi}\right)_4 (a^2+b^2)^{-z}. \end{aligned}$$

Here also is the corresponding formula for L-series coefficients c_p: when $p=2$, $p|n$ or $p\equiv 3 \bmod 4$, we have $c_p=0$; otherwise [14]

$$c_p = 2\left(\frac{2}{p}\right)\cdot\begin{cases} -a & \text{if } n^{(p-1)/4}\equiv 1 \bmod p, \\ a & \text{if } n^{(p-1)/4}\equiv -1 \bmod p, \\ -b & \text{if } n^{(p-1)/4}b\equiv -a \bmod p \\ b & \text{if } n^{(p-1)/4}b\equiv a \bmod p \end{cases}$$

where $p=a^2+b^2$ with $a\equiv 3 \bmod 4$, $b\equiv 0 \bmod 2$. Again, the sequence of integers for which $y^2=x^3-n\,x$ has zero rank [15]:

$$\ldots, -12, -11, -10, -7, -6, -4, -2, -1, 1, 3, 4, 8, 9, 11, 13, 18, \ldots$$

is worthy of deeper study.

As a quintic follow-on to [5, 7], we merely mention [16].

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