Soap Film Experiments

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We conclude our brief survey of minimal surfaces, started in [1, 2], with more solutions of Plateau's problem. The functions $F[\phi, m]$ and $K[m]$ are defined exactly as before.

0.1. Ramp Inside a Cube. Consider a polygonal wire loop with six line segments:

$$(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1) \to (0,1,1) \to (0,1,0) \to (0,0,0).$$

What is the minimal area for any surface spanning this fixed boundary? Equivalently, what is the outcome of dipping the wire loop in a soap solution? Following [3, 4, 5], we numerically solve the equation

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2 + \sqrt{2 - \lambda}}} \frac{K\left[\dfrac{8\sqrt{2 - \lambda}}{\left(2 + \sqrt{2 - \lambda}\right)^2}\right]}{K\left[\dfrac{2 - \sqrt{2 - \lambda}}{2 + \sqrt{2 - \lambda}}\right]}$$

and obtain $\lambda = 1.5733414653....$ Define

$$
\begin{aligned}
x(u, v) &= \kappa \operatorname{Re} \int\limits_{0}^{u+iv} \frac{1 - \tau^2}{\sqrt{1 + \lambda\, \tau^4 + \tau^8}}\, d\tau \\
&= \kappa \operatorname{Re} \left\{ \frac{F\left[\arcsin\left(\sqrt{2 + \sqrt{2 - \lambda}}\, \dfrac{\omega}{1 + \omega^2}\right), \dfrac{2 - \sqrt{2 - \lambda}}{2 + \sqrt{2 - \lambda}}\right]}{\sqrt{2 + \sqrt{2 - \lambda}}} \right\},
\end{aligned}
$$

$$y(u,v) \;=\; \kappa \operatorname{Re} \int_0^{u+iv} \frac{i(1+\tau^2)}{\sqrt{1+\lambda\,\tau^4+\tau^8}}\, d\tau$$

$$=\; \kappa \operatorname{Re}\left\{ \frac{F\left[\arcsin\left(\sqrt{2+\sqrt{2-\lambda}}\,\frac{i\,\omega}{1-\omega^2}\right), \dfrac{2-\sqrt{2-\lambda}}{2+\sqrt{2-\lambda}}\right]}{\sqrt{2+\sqrt{2-\lambda}}}\right\},$$

$$z(u,v) \;=\; \kappa \operatorname{Re} \int_0^{u+iv} \frac{2\tau}{\sqrt{1+\lambda\,\tau^4+\tau^8}}\, d\tau$$

$$=\; \sqrt{2}\,\kappa \operatorname{Re}\left\{ \frac{F\left[\arcsin\left(\sqrt{\dfrac{-\lambda+\sqrt{-4+\lambda^2}}{2}}\,\omega^2\right), \dfrac{\left(\lambda+\sqrt{-4+\lambda^2}\right)^2}{4}\right]}{\sqrt{-\lambda+\sqrt{-4+\lambda^2}}}\right\}$$

where the complex line integrals have endpoint $\omega = u + iv$ satisfying

$$u^2 + v^2 \le 1, \qquad |v| \ge u$$

– call this planar domain Ω – and the normalization constant κ satisfies

$$\frac{1}{\kappa} = 2\sqrt{2}\operatorname{Re}\left\{ \frac{1}{\sqrt{-\lambda+\sqrt{-4+\lambda^2}}} K\left[\frac{\left(\lambda+\sqrt{-4+\lambda^2}\right)^2}{4}\right]\right\}.$$

These expressions give the top portion ($z > 0$) of the surface in Figures 1 and 2. A reflection provides the bottom portion; a rotation would further align the surface with our six prescribed vertices. This is a representative of the *Schwarz CLP* family of minimal surfaces; a nice contrast exists with the Schwarz D surface [6]. We also have surface area

$$2\iint_\Omega \sqrt{eg - f^2}\, dv\, du = 1.7816507345...$$

where e, f, g are as in [2]. Brakke and Weber duplicated this calculation, using Surface Evolver software [7] and conformal mapping techniques [8] respectively.
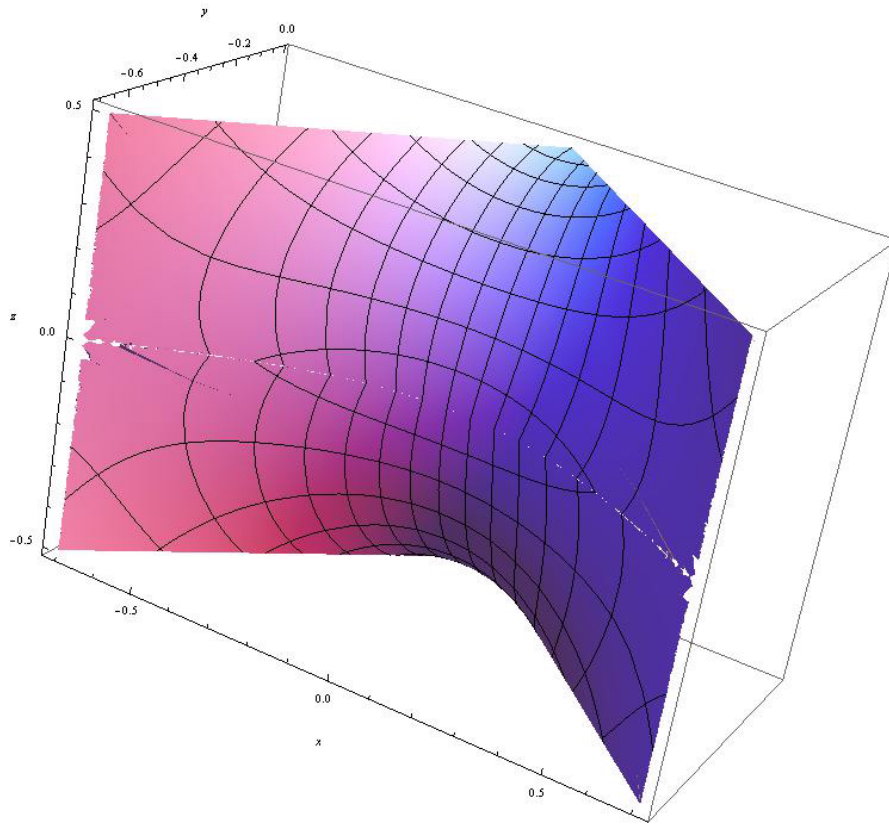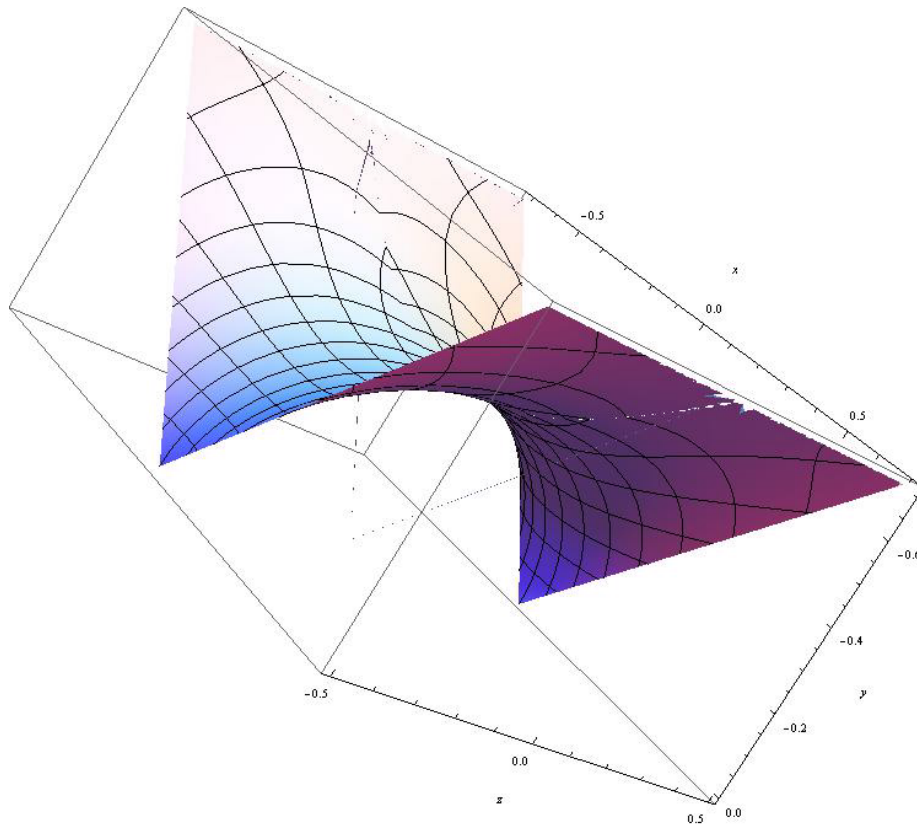
Figure 1: First view of CLP surface

Figure 2: Second view of CLP surface

0.2. Saddle Inside a Cube. Consider a polygonal wire loop with eight line segments:

$$(0,0,1) \to (1,0,1) \to (1,0,0) \to (1,1,0) \to (1,1,1) \to (0,1,1) \to (0,1,0) \to (0,0,0) \to (0,0,1).$$

Again, what is the minimal area for any surface spanning this fixed boundary? Following [9], we numerically solve the equation

$$\frac{1}{2} = \sqrt{\frac{\sqrt{2-\lambda}}{-\lambda + \sqrt{-4+\lambda^2}}} \frac{K\left[\frac{\left(\lambda + \sqrt{-4+\lambda^2}\right)^2}{4}\right]}{K\left[\frac{1}{2} - \frac{1}{\sqrt{2-\lambda}}\right]}$$

and obtain $\lambda = -5.3485781991....$ Define

$$
\begin{aligned}
x(u,v) &= \kappa \operatorname{Re} \int_0^{u+iv} \frac{1-\tau^2}{\sqrt{1+\lambda\,\tau^4+\tau^8}}\, d\tau \\
&= \kappa \operatorname{Re} \left\{ \frac{F\left[\arcsin\left(\sqrt{\frac{2\sqrt{2-\lambda}}{1+\sqrt{2-\lambda}\,\omega^2+\omega^4}}\,\omega\right), \frac{1}{2} - \frac{1}{\sqrt{2-\lambda}}\right]}{\sqrt{2\sqrt{2-\lambda}}} \right\},
\end{aligned}
$$

$$
\begin{aligned}
y(u,v) &= \kappa \operatorname{Re} \int_0^{u+iv} \frac{i(1+\tau^2)}{\sqrt{1+\lambda\,\tau^4+\tau^8}}\, d\tau \\
&= \kappa \operatorname{Re} \left\{ \frac{i\,F\left[\arcsin\left(\sqrt{\frac{2\sqrt{2-\lambda}}{1+\sqrt{2-\lambda}\,\omega^2+\omega^4}}\,\omega\right), \frac{1}{2} + \frac{1}{\sqrt{2-\lambda}}\right]}{\sqrt{2\sqrt{2-\lambda}}} \right\},
\end{aligned}
$$

$$z(u,v) = \kappa \operatorname{Re} \int_0^{u+iv} \frac{2\tau}{\sqrt{1 + \lambda\, \tau^4 + \tau^8}}\, d\tau$$

$$= \sqrt{2}\kappa \operatorname{Re} \left\{ \frac{F\left[\arcsin\left(\sqrt{\dfrac{-\lambda + \sqrt{-4 + \lambda^2}}{2}}\, \omega^2\right), \dfrac{\left(\lambda + \sqrt{-4 + \lambda^2}\right)^2}{4}\right]}{\sqrt{-\lambda + \sqrt{-4 + \lambda^2}}} \right\}$$

where the complex line integrals have endpoint $\omega = u + iv$ satisfying

$$u^2 + v^2 \leq 1, \qquad |v| \geq u$$

– call this planar domain Ω – and the normalization constant κ satisfies

$$\frac{1}{\kappa} = \frac{2\sqrt{2}}{\sqrt{-\lambda + \sqrt{-4 + \lambda^2}}} K\left[\frac{\left(\lambda + \sqrt{-4 + \lambda^2}\right)^2}{4}\right].$$

(No call to the Re function is needed here, unlike before.) These expressions give a quarter-wedge of the surface in Figures 3 and 4. Reflections provide the other three quarter-wedges; a rotation would further align the surface with our eight prescribed vertices. This is a representative of the *Schwarz T* family of minimal surfaces, also known as *tD surfaces* (generalizing the D surface). We finally have surface area

$$4 \iint_\Omega \sqrt{eg - f^2}\, dv\, du = 2.4674098291... = 2(1.2337049145...),$$

duplicating a calculation by Brakke [7]. The CLP expression for z is identical to the T expression for z; this is true for x and y too (although less apparently so). The latter expressions for $\{x, y\}$ give elliptic parameters $\{1/4, 3/4\}$ when $\lambda = -14$, consistent with our earlier work [2]. The former expressions, which come from [3], give $\{-1/3, -1/3\}$ instead. Yet another set of expressions appear in [9], which we have not attempted to use.

The presence of the constant $1.2337049145...$, which also appeared in [1], indicates that the T surface is related to Gergonne's surface [9, 10]. This is surprising because the T surface is the solution of a fixed boundary problem whereas Gergonne's surface solves a problem involving a partially free boundary.
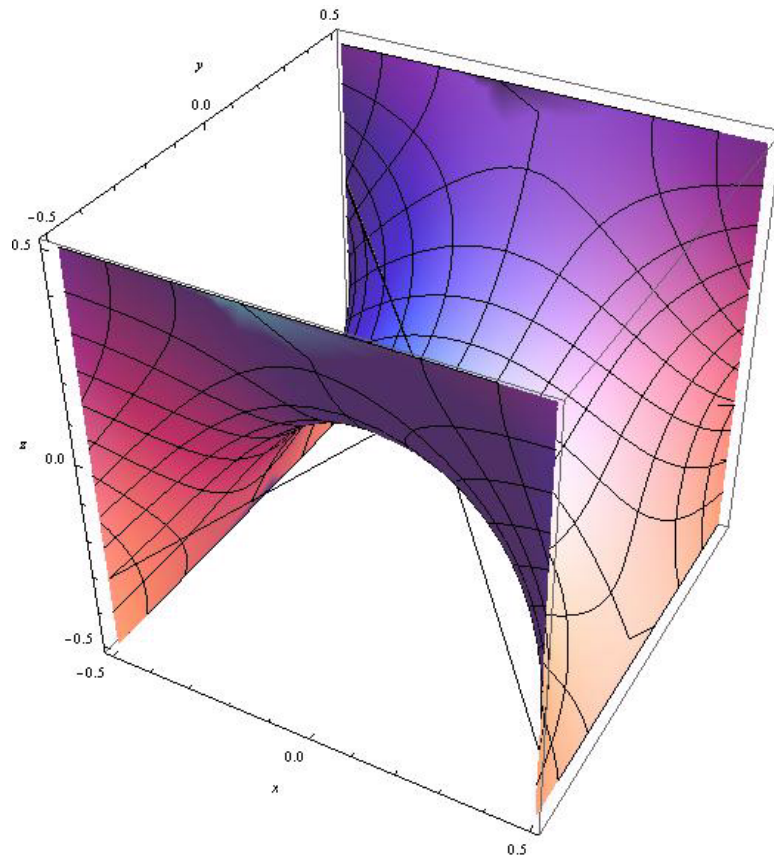
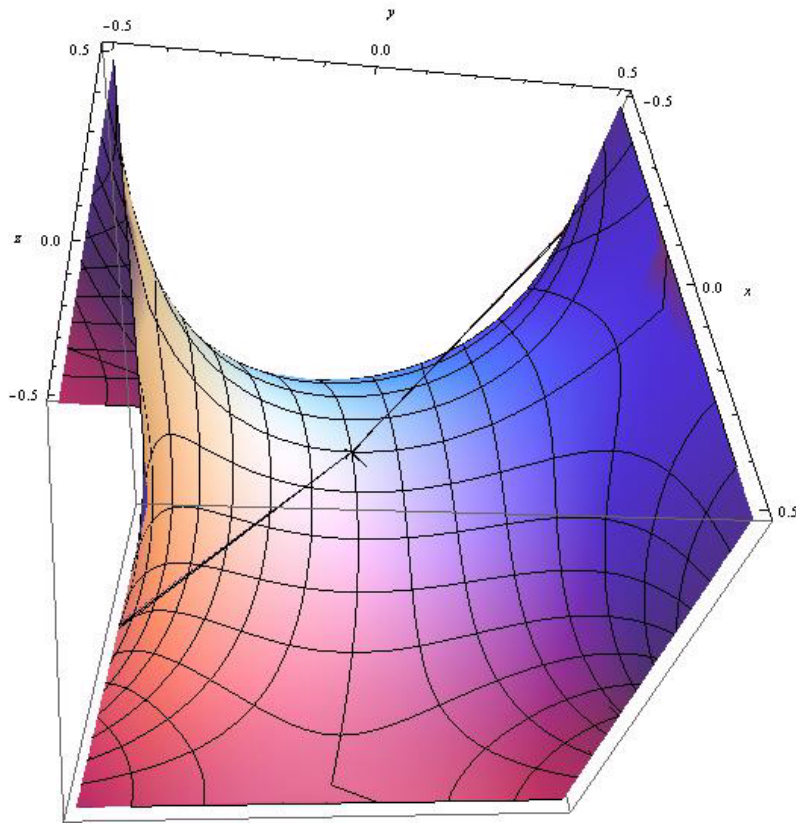Figure 3: First view of T surface
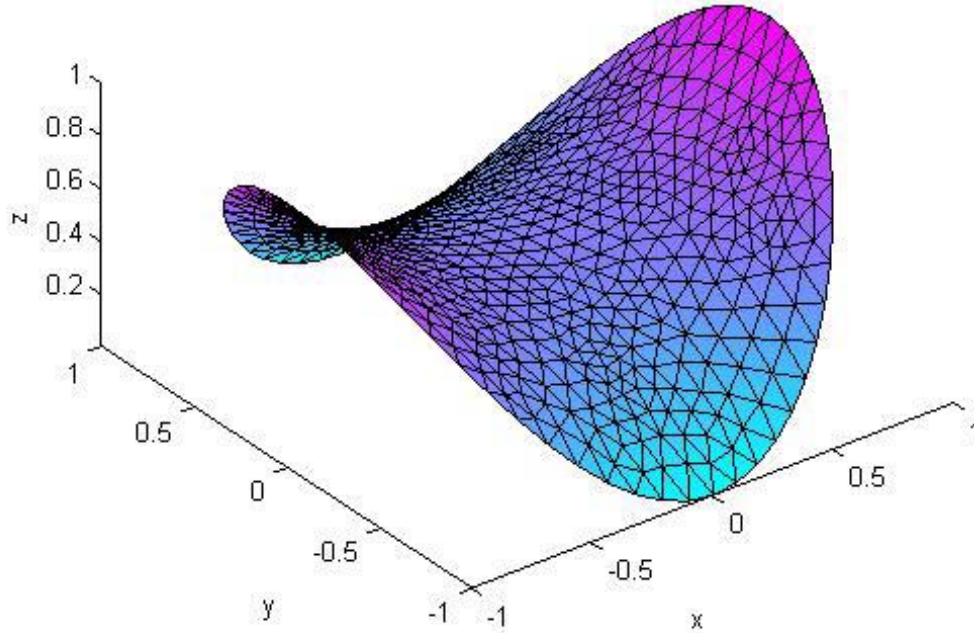
Figure 4: Second view of T surface

Figure 5: Surface from Matlab help pages with boundary C

0.3.   Other Problems.   Consider a smooth wire loop C given parametrically by

$$x = \cos(\theta), \quad y = \sin(\theta), \quad z = \cos(\theta)^2, \quad 0 \le \theta < 2\pi.$$

The projection of C into the xy-plane is the unit circle; its projection into the xz-plane is the parabola $z = x^2$; its projection into the yz-plane is the parabola $z = 1 - y^2$. The arclength of C is

$$4\sqrt{2}E\left[\frac{1}{2}\right] = 7.6403955780... = 4\,(1.9100988945...) > 2\pi$$

which incidentally is the arclength of the planar sine curve (one period). A closed-form expression for the area $3.8269736664... > \pi$ of the minimal surface spanning C is unknown [11, 12]. See Figure 5.
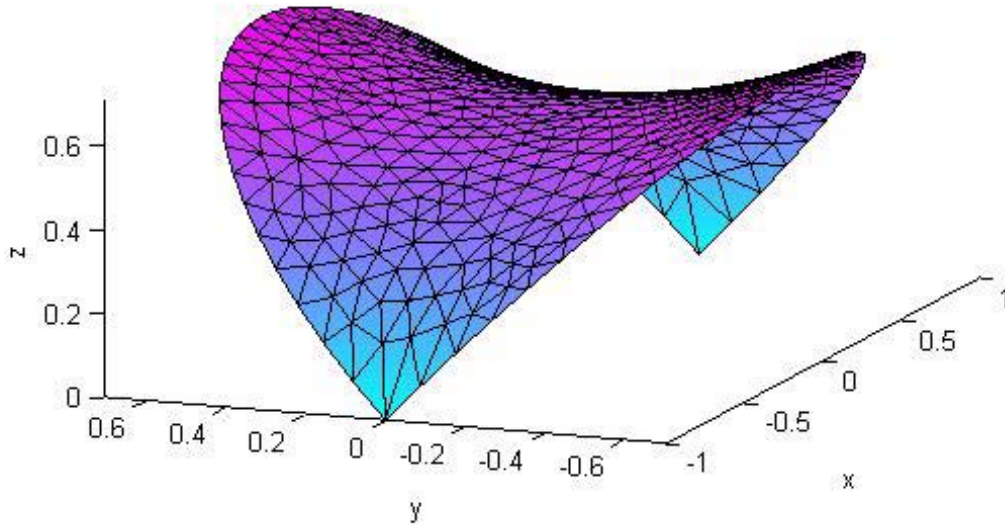
Figure 6: Surface spanning folded circular loop

Consider instead the folded circular loop, that is, the outcome of orthogonally mounting two unit semicircles along common diameters. For the boundary configuration shown in Figure 6, we deduce that its projection in the xy-plane is the ellipse $x^2 + 2y^2 = 1$ and its height z is simply $|y|$. The arclength is obviously 2π; the surface area $2.4822844847... < (2 + \pi)/2$ is again unknown [13, 14].

We wonder finally what can be said about minimal surfaces that span three disjoint perpendicular cubic edges. This topic is believed to be more difficult than the "two diagonals" analog (Gergonne's surface) and relevant help would be appreciated.

References

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