Volumes of Hyperbolic 3-Manifolds

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Hyperbolic n-space is the n-dimensional real upper half-space

$$\mathbb{H}^n = \{\xi \in \mathbb{R}^n : x_n > 0\}, \qquad \xi = (x_1, x_2, x_3, \ldots, x_n),$$

endowed with the complete Riemannian metric $ds = |d\xi|/x_n$ of constant sectional curvature equal to -1. That is, the geodesics of $\mathbb{H}^n$ consist entirely of semicircles and vertical lines that are orthogonal to the $(n-1)$-dimensional boundary $\mathbb{R}^{n-1} \times \{0\}$.

A **hyperbolic n-manifold** M is an n-dimensional connected manifold with a complete Riemannian metric such that every point of M has a neighborhood isometric with an open subset of $\mathbb{H}^n$ [1]. Such a manifold may be either orientable or nonorientable. It is **open** if it has at least one cusp, for example, a puncture in $n = 2$ (see Figures 1 and 2); otherwise it is **closed**.

From the notion of length along a geodesic proceeds the definition of volume $\mathrm{vol}(M)$ of a hyperbolic manifold. Unlike the Euclidean case, this is an important characteristic of M. If two finite-volume hyperbolic n-manifolds are homeomorphic, where $n \geq 3$, then they must be isometric. This surprising fact (false for $n = 2$) is known as the Mostow-Prasad rigidity theorem [2, 3] and is believed to be crucial for the classification of 3-manifolds. We henceforth restrict attention only to manifolds with finite volume; the topological invariance of $\mathrm{vol}(M)$ follows from the Gauss-Bonnet theorem when $n = 2$ and via Mostow-Prasad rigidity when $n \geq 3$.

Define the **volume spectrum** $\mathrm{spc}(n)$ to be the set of all volumes of finite-volume hyperbolic n-manifolds. It is known that [4, 5]

$$\mathrm{spc}(2) = \{2\pi k : k \geq 1\}, \qquad \mathrm{spc}(4) = \left\{\frac{4\pi^2}{3}k : k \geq 1\right\}$$

but $\mathrm{spc}(3)$ is far more complicated. Let us restrict attention only to orientable 3-manifolds and call the consequential subset $\mathrm{spc}_{\mathrm{o}}(3)$. Let ω denote the first infinite ordinal. Gromov, Jørgensen and Thurston [6, 7, 8] proved that $\mathrm{spc}_{\mathrm{o}}(3)$ is a closed, non-discrete, well-ordered set of positive real numbers which looks like

$$\begin{aligned} v_1 &< v_2 < v_3 < \ldots < v_\omega < v_{\omega+1} < v_{\omega+2} < \ldots < v_{2\omega} < v_{2\omega+1} < \ldots \\ &< v_{3\omega} < v_{3\omega+1} < \ldots < v_{\omega^2} < v_{\omega^2+1} < \ldots < v_{\omega^3} < v_{\omega^3+1} < \ldots \end{aligned}$$

where

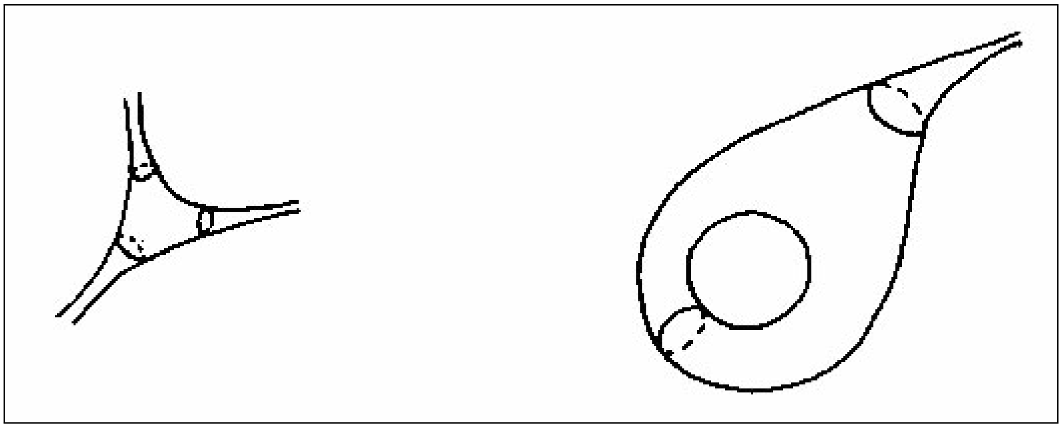

Figure 1: There exist two orientable surfaces with hyperbolic volume 2π: a sphere with 3 punctures and a torus with 1 puncture.

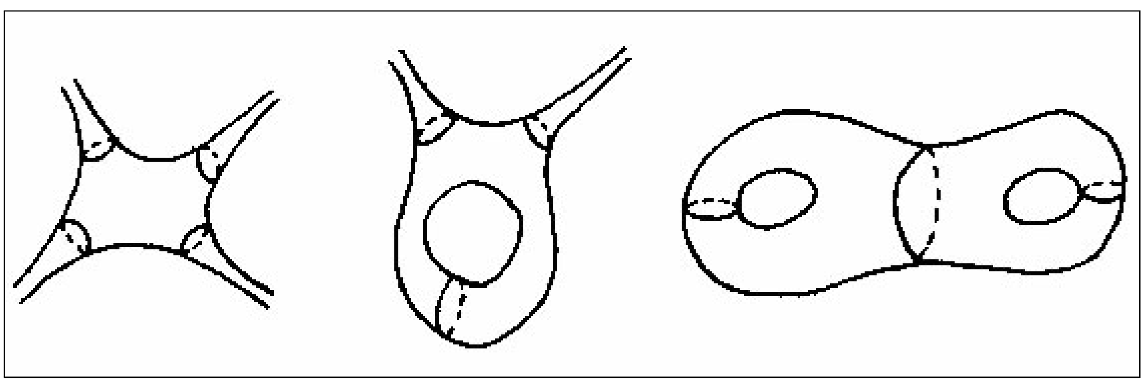

Figure 2: There exist three orientable surfaces with hyperbolic volume 4π: a sphere with 4 punctures, a torus with 2 punctures, and a (closed) connected sum of two tori.

- v_1 is the least volume of a closed orientable 3-manifold,
- v_2 is the next smallest volume of a closed orientable 3-manifold,
- $v_\omega = \lim_{k\to\infty} v_k$ is the least volume of an (open) orientable 3-manifold with one cusp and is the first limit point in $\mathrm{spc}_\mathrm{o}(3)$,
- $v_{2\omega} = \lim_{k\to\infty} v_{\omega+k}$ is the next smallest volume of an (open) orientable 3-manifold with one cusp and is the second limit point in $\mathrm{spc}_\mathrm{o}(3)$,
- $v_{\omega^2} = \lim_{k\to\infty} v_{k\omega}$ is the least volume of an (open) orientable 3-manifold with two cusps and is the first limit point of limit points in $\mathrm{spc}_\mathrm{o}(3)$.

The set $\mathrm{spc}_\mathrm{o}(3)$ is said to have ordinal type ω^ω. For convenience, we will henceforth use the phrase "minimal manifold" to refer to a "least-volume manifold".

Weeks [9] and Matveev & Fomenko [10] independently discovered what is conjectured to be the unique minimal closed orientable 3-manifold. It has volume given by [11, 12, 13]

$$v_1 = \mathrm{Im}\left[\mathrm{Li}_2(z_0) + \ln(|z_0|)\ln(1-z_0)\right] = 0.9427073627...$$

where

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\ln(1-u)}{u}du, \qquad |z| \le 1$$

is the dilogarithm function [14] and z_0 is the zero of the cubic $z^3 - z^2 + 1$ with $\mathrm{Im}(z) > 0$. Evidence supporting this conjecture includes [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]; the best known rigorous lower bound $v_1 \ge 0.324$ can be strengthened to $v_1 \ge 0.547$ [31] if Perelman's proof of the Poincaré conjecture is confirmed. The next smallest volume is conjectured to be $v_2 = 0.9813688288...$ [32]. Cao & Meyerhoff [33] proved that there exist two minimal 1-cusped orientable 3-manifolds; one of the manifolds is the complement of the figure-eight knot [34, 35] in $\mathbb{H}^3$ and has volume given by

$$\begin{aligned} v_\omega &= 2\,\mathrm{Im}\left[\mathrm{Li}_2(e^{i\pi/3})\right] = 2\,\mathrm{Cl}_2(\pi/3) = 3\,\mathrm{Cl}_2(2\pi/3) \\ &= \frac{9\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{2n+1}{(3n+1)^2(3n+2)^2} \\ &= 2(1.0149416064...) = 2.0298832128..., \end{aligned}$$

where Clausen's integral is defined by

$$\mathrm{Cl}_2(\theta) = \sum_{k=1}^{\infty}\frac{\sin(k\theta)}{k^2} = -\int_0^\theta \ln\left(2\sin(\frac{t}{2})\right)dt = \mathrm{Im}\left[\mathrm{Li}_2(e^{i\theta})\right].$$

Broadhurst [36, 37, 38] found a series that can be used as a base-3 digit-extraction algorithm for v_ω:

$$v_\omega = \frac{2\sqrt{3}}{9}\sum_{n=0}^{\infty}\frac{(-1)^n}{27^n}\left(\frac{9}{(6n+1)^2}-\frac{9}{(6n+2)^2}-\frac{12}{(6n+3)^2}-\frac{3}{(6n+4)^2}+\frac{1}{(6n+5)^2}\right).$$

Define $L = v_\omega/2 = 1.0149416064...$ [39] to be **Lobachevsky's constant**, which we will need later. The next smallest volume of a 1-cusped orientable 3-manifold is conjectured to be $v_{2\omega} = 2.5689706009...$ [40, 41]. Finally, it is conjectured that the Whitehead link complement is a minimal 2-cusped orientable 3-manifold, which has volume given by [42]

$$v_{\omega^2} = 4\,\mathrm{Cl}_2(\pi/2) = 4G = 3.6638623767...$$

where G is Catalan's constant [43, 44]. Much more about $\mathrm{spc}_{\mathrm{o}}(3)$ still awaits discovery.

The full set $\mathrm{spc}(n)$ is well-ordered but surprisingly different from $\mathrm{spc}_{\mathrm{o}}(3)$. The minimal closed nonorientable 3-manifold appears to have volume $2L$ (the same as the figure-eight complement) [32], but the minimal 1-cusped nonorientable 3-manifold was proved by Adams [45, 46] to be what is called the Gieseking manifold, which has volume L (only half as large). The next smallest volume of a 1-cusped nonorientable 3-manifold is conjectured to be 1.8319311884.... It is known that $2L$ is also the volume of the minimal 2-cusped nonorientable 3-manifold [47].

The complement of a knot in $\mathbb{H}^3$ admits a hyperbolic structure unless it is a torus or satellite knot. Automated techniques [48] exist for computing volume and other hyperbolic invariants of 3-manifolds, which serve to distinguish knots up to homeomorphism [49, 50, 51, 52, 53]. The so-called "volume conjecture" relates, for any knot, the asymptotic behavior of its colored Jones polynomial evaluated at a root of unity to its volume [11, 54].

We now generalize. A **Kleinian group** is a discrete nonelementary subgroup of the group of all orientation-preserving isometries of $\mathbb{H}^3$. A **hyperbolic 3-orbifold** is a quotient of $\mathbb{H}^3$ by a Kleinian group, possibly with torsion. (An orientable 3-manifold is a special case of a 3-orbifold for which the Kleinian group is torsion-free.) The volume spectrum $\mathrm{spc}_{\mathrm{o}}'(3)$ of orientable 3-orbifolds is of ordinal type ω^ω [55] and is quite similar to before, where

- v_1' is the least volume of a closed orientable 3-orbifold,
- $v_{l\omega}' = \lim_{k\to\infty} v_{(l-1)\omega+k}'$ is the l^{th} limit point in $\mathrm{spc}_{\mathrm{o}}'(3)$, where $l = 1, 2, 3, ...$

The unique minimal closed orientable 3-orbifold is conjectured to have volume [56, 57, 58]

$$v_1' = \frac{1}{60}\sum_{j=1}^{3}\mathrm{Im}\left[\mathrm{Li}_2(z_j) + \ln(|z_j|)\ln(1-z_j)\right] = 0.0390502856...$$

where z_1 is the zero of the quartic $z^4 - 2z^3 + z - 1$ with $\mathrm{Im}(z) > 0$, and z_2, z_3 are the two distinct zeroes of the octic $z^8 - 3z^7 + 5z^6 - 5z^5 + 3z^4 - z + 1$ satisfying both $\mathrm{Re}(z) < 1$ and $0 < \mathrm{Im}(z) < 1$. See [16, 59, 60, 61, 62] for supporting evidence. Unlike what occurs for orientable manifolds, however, the volume u' of the minimal 1-cusped orientable 3-orbifold is not equal to the limit point v'_ω. Adams [63] and Meyerhoff [16, 64] proved that

$$u' = L/12 = 0.0845784672... < v'_\omega = G/3 = 0.3053218647....$$

In fact [65, 66, 67], the six open orientable orbifolds of volume less than $L/4$ have volumes $L/12$, $G/6$, $L/6$, $L/6$, $5L/24$, and $G/4$, whereas

$$v'_{2\omega} = \frac{7}{24}\left[\mathrm{Cl}_2\left(\frac{2\pi}{7}\right) + \mathrm{Cl}_2\left(\frac{4\pi}{7}\right) - \mathrm{Cl}_2\left(\frac{6\pi}{7}\right)\right] = 0.4444574639...$$

$$v'_{3\omega} = \frac{G}{2} = 0.4579827970...$$

See [13, 57] for an interesting unsolved problem about linear relations involving Clausen function values. Finally [65], with regard to the full set $\mathrm{spc}'(3)$, the six open nonorientable orbifolds of volume less than $L/8$ have volumes $L/24$, $G/12$, $L/12$, $L/12$, $5L/48$, and $G/8$. The minimal closed nonorientable 3-orbifold appears not to be known. A remarkable connection between shortest geodesic lengths in closed arithmetic 3-orbifolds and Lehmer's conjecture from number theory [68] is described in [1, 69, 70].

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